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LETTER TO THE EDITOR

Conformal tensor and Petrov classification for the bounded part of the Liénard–Wiechert field

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Abstract. The existing analogies between the Liénard–Wiechert electromagnetic field (LWF) produced by a moving charge and the Robinson–Trautman solutions (RT) of general relativity are further enlarged by showing that the Petrov type of the bounded part of an LWF is the same as that of a generic RT solution. We argue that it is possible to define an electromagnetic conformal tensor by using a superpotential previously found for the LWF. The conformal tensor so constructed allows us to 'Petrov-classify' that part of the LWF.

Analogies between classical electrodynamics and general relativity have been known for a long time. It is known, for example, that gravitational effects may be formally described as if they were electromagnetic phenomena in a medium (Synge 1960, Landau and Lifshitz 1975) and, correspondingly, that electromagnetic phenomena in non-magnetic media may also be formally described using vacuum Maxwell equations defined on a curved manifold endowed with a certain metric (Núñez-Yépez *et al* 1988). Analogies of this sort have been studied by Newman (1974) who showed that there are interesting similarities between the Liénard–Wiechert field (LWF) produced by an accelerated charge and the Robinson–Trautman metrics (RT) of general relativity (Robinson and Trautman 1962). Our purpose in this work is to exhibit a further algebraic analogy between the LWF and RT. We show that it is possible to define an electromagnetic conformal (or Weyl) tensor C_{ijkl} for the bounded part of an LWF which is generally of type II in the Petrov classification, as is precisely the case with the most general conformal tensor associated with RT solutions in general relativity (Kramer *et al* 1980, Synge 1964), although we should mention that RT solutions can also be specialized to Petrov types D, III, N or O. Moreover, we might say that the non-analogous features found between the LWF and the RT solutions have to do with the different physical and mathematical structures of the Maxwell and the Einstein fields. The radiative field in the LW solutions is perhaps the main source of differences between the LWF and RT metrics. However incomplete, the importance of such analogies has been pinpointed by Newman (1974).

A charged particle moving arbitrarily in Minkowski 4-space produces at points x^r on its forward light-cone (figure 1) a retarded LW electromagnetic field (LWF) which in Heaviside–

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Lorentz units with $c = 1$ is

$$F^{ab}(x^r; z^r(\tau)) = \frac{q}{R^2}\left(k^{[a}\ddot{z}^{b]} + \frac{(1-W)}{R}k^{[a}\dot{z}^{b]}\right) = \frac{q}{R^2}U^{[a}k^{b]} \tag{1}$$

where q is the particle's electric charge, $z^r(\tau)$ stands for the particle's position on its world-line C as a function of the proper-time τ, $v^b = \mathrm{d}z^b/\mathrm{d}\tau$ is the 4-velocity, $a^b = \mathrm{d}v^b/\mathrm{d}\tau$ is the 4-acceleration, $k^r \equiv x^r - z^r(\tau)$ so that k^r is null: $k_i k^i = 0$, $R \equiv -k^r v_r$ is the retarded distance from P to Q, $W \equiv -k^r a_r$, $B \equiv (1 - W)/R$ is the Plebański invariant (Plebański 1972), $U^r = Bv^r + a^r$ is the Synge vector (Synge 1970), and $X_{[ab]} \equiv X_{ab} - X_{ba}$ stands for antisymmetrization. The first term on the right-hand side of the first line of (1) is the radiative (R) part of the field in the sense that its energy flux is non-vanishing even at large distances to the charge, whereas the second term can be regarded as bounded (B) to the moving point-charge. Such splitting in a sum of terms, one radiative and another bounded (Teitelboim 1970), is important for the rest of our discussion.

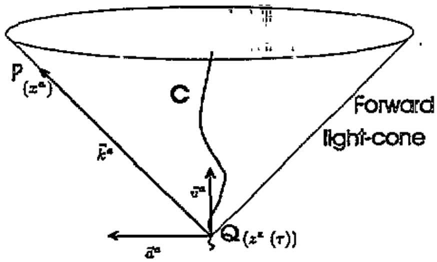

Figure 1. Kinematics of the world-line C of the point-charge emitting the Liénard–Wiechert field. The field-point x^r (P) is on the forward light-cone, $z^r(\tau)$ (Q) is the charge's position on the world-line, $v^b = \mathrm{d}z^b/\mathrm{d}\tau$ is the 4-velocity, $a^b = \mathrm{d}v^b/\mathrm{d}\tau$ is the 4-acceleration, $k^r \equiv x^r - z^r(\tau)$ so k^r is null, $R \equiv -k^r v_r$ is the retarded distance from P to Q, $B \equiv (1 + k^r a_r)/R$ and $U^r \equiv Bv^r + a^r$.

The information about the electromagnetic field energy and momentum is contained in a symmetric Maxwell tensor $T^{ab} = F^a_c F^{cb} - (F_{ij}F^{ij})\eta^{ab}$, where $\eta^{ab} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric. In the case of LWF, this tensor also splits into a radiative and a bounded part (Teitelboim 1970, van Weert 1974, Gaftoi *et al* 1993) as follows

$$T^{ab} = T^{ab}_{\mathrm{R}} + T^{ab}_{\mathrm{B}} \tag{2}$$

where

$$T^{ab}_{\mathrm{R}} = \frac{q^2}{R^4}(a^r a_r - R^{-2}W^2)k^a k^b \tag{3a}$$

and

$$T^{ab}_{\mathrm{B}} = \frac{q^2}{R^4}\left(\frac{\eta^{ab}}{2} + k^{(a}a^{b)} + Bk^{(a}v^{b)} - R^{-2}(1-2W)k^a k^b\right) \tag{3b}$$

where $X_{(ab)} = X_{ab} + X_{ba}$ stands for symmetrization. These two parts of the energy–momentum tensor are dynamically independent in the sense that each one satisfies separately the continuity equation

$$T^{ab}_{\mathrm{R},b} = T^{ab}_{\mathrm{B},b} = 0. \tag{4}$$

It has been shown (van Weert 1974) that (4) implies the existence of a super-potential K_{abc} generating the Maxwell energy–momentum tensor according to

$$T^{ab} = K^{ajb}_{\ \ ,j}. \tag{5}$$

We are only interested in the bounded part of the energy–momentum tensor so, from now on and for the sake of simplicity, we remove the subscript B from all the tensors as the bounded part is the only one in which we are interested. The expression for the superpotential generating the bounded part of the energy–momentum tensor of LWF is (van Weert 1974, Aquino *et al* 1993, López-Bonilla *et al* 1994, Gaftoi *et al* 1994)

$$K_{bjc} = \frac{q^2}{4R^4}\left(\frac{1}{R}(4W-3)v_{[b}k_{j]}k_c - 4a_{[b}k_{j]}k_c + \eta_{cj}k_b - \eta_{cb}k_j\right) \tag{6}$$

which obviously fulfils equation (5).

An analysis of the superpotential (6) carried out in Newman–Unti coordinates (Aquino *et al* 1993) has shown that both the algebraic and the differential properties of K_{abc} are identical to those of the so-called Lanczos spin-tensor introduced in general relativity (Lanczos 1962, Ares-de-Parga *et al* 1989, López-Bonilla *et al* 1993). These features hint toward taking advantage of the similarities between K_{abc} and the spin-tensor for defining an electromagnetic Weyl tensor which may allow a Petrov classification of the bounded part of the LWF to be carried out. We do this later. However, before doing that, let us point out that the radiative part of an LWF can also be generated by an appropriate superpotential which, however, has not the structure of a spin-tensor. Thence, it cannot be used to generate an electromagnetic conformal tensor or to 'Petrov-classify' the radiative part of an LWF.

The superpotential K_{abc} previously introduced for generating the bounded part of the energy–momentum tensor for the LWF has been shown to be an intrinsic angular momentum density for the electromagnetic field of a moving charge (Ares-de-Parga *et al* 1990). From the explicit expression for K_{abc} given in (6), it is simple to check that the superpotential has the following properties:

$$K_{abc} = -K_{bac} \tag{7a}$$

$$K_{abc} + K_{bca} + K_{cab} = 0 \tag{7b}$$

$$K_{a\ \ b}^{\ b} = 0 \tag{7c}$$

and

$$K_{ab\ \ ,c}^{\ \ c} = 0. \tag{7d}$$

These properties are important for our discussion since they are identical (Lanczos 1962, López-Bonilla *et al* 1993) to those fulfilled by the spin-tensor associated with solutions of Einstein field equations—and so defined on curved manifolds. The only proviso we need is to interpret the covariant derivatives required by the spin-tensor as the ordinary partial derivatives appearing in (7). It is now natural to regard K_{ijk} as a sort of spin-tensor for the LWF and use it as the generator of an electromagnetic conformal tensor C_{ijkl} appropriate for the problem. This use of the superpotential may be further justified in the already pinpointed fact that K_{ijk} is related to the intrinsic angular momentum of the charge's field. We may then use the relationship between the spin-tensor and the tensor C_{ijkl} in the form valid on a curved manifold to construct a conformal tensor for the bounded part of an LWF:

$$C_{jrim} = K_{jri,m} - K_{jrm,i} + K_{imj,r} - K_{imr,j} + \eta_{jm}K_{i\ \ r,a}^{\ a} - \eta_{ij}K_{m\ \ r,a}^{\ a} + \eta_{ri}K_{m\ \ j,a}^{\ a} - \eta_{rm}K_{i\ \ j,a}^{\ a} \tag{8}$$

where according to (4), $K_r{}^a{}_{j,a} = T_{rj}$. We can thus regard the tensor

$$C_{ijrb} = \eta_{i[b}T_{r]j} + \eta_{j[r}T_{b]i} - \frac{4}{R}K_{rb[i}R_{,j]} - \frac{4}{R}K_{ij[r}R_{,b]}$$
$$-\frac{q^2}{2R^4}\Bigg[\eta_{i[r}\eta_{b]j} + 2B(v_r\eta_{b[j}k_{i]} - v_b\eta_{r[j}k_{i]})$$
$$+\frac{2}{R}k_{[i}a_{j]}v_{[r}k_{b]} + 2(a_r\eta_{b[j}k_{i]} - a_b\eta_{r[j}k_{i]})\Bigg]$$
$$-\frac{q^2}{2R^4}\Bigg[\frac{2}{R}k_{[r}a_{b]}v_{[i}k_{j]} + 2(a_i\eta_{j[b}k_{r]} - a_j\eta_{i[b}k_{r]}) + \eta_{r[i}\eta_{j]b}$$
$$+2B(v_i\eta_{j[b}k_{r]} - v_j\eta_{i[b}k_{r]})\Bigg] \qquad (9)$$

as the electromagnetic Weyl tensor associated with the LWF. It is now straightforward to analyse C_{ijkl} to conclude (Synge 1964, Kramer *et al* 1980) that for the bounded field produced by a point-charge in arbitrary motion, C_{ijkl} is *of type II in the Petrov classification* in analogy with the general case of RT solutions in general relativity. In this general case there are two non-degenerate null Debever–Penrose vectors which could be important for studying other properties of the LWF. If, however, the 4-acceleration vanishes but *not* the 3-velocity, v, then C_{ujkl} *is of type D in the Petrov classification.* In the two cases analysed (i.e. arbitrary or vanishing 4-acceleration with $v \neq 0$), the null 4-vector $k^r (\equiv x^r - z^r(\tau))$ is a doubly-degenerate Debever–Penrose vector (Synge 1964, Kramer *et al* 1980); this follows from the relation

$$C_{jri[m}k_{n]}k^r k^i = 0. \qquad (10)$$

Therefore, k^r must point toward the principal null-direction of the bounded parts of the Faraday and Maxwell tensors associated with the LWF:

$$F^{ab}k_b = \frac{q}{R^2}k^a \qquad T^{ab}k_b = \frac{q^2}{2R^4}k^a. \qquad (11)$$

Finally, *if both the 4-acceleration and the 3-velocity vanish,* $a^r = v = 0$, *then* $C_{ijkl} = 0$, *i.e. a Coulomb field is of type O in the Petrov classification.*

What happens with the radiative part of the LWF? For such a part, we cannot construct a conformal tensor using the analogy with the spin-tensor and therefore it cannot be Petrov-classified. We may say, however, that using the expression for F_{R}^{ab} given in (1), it is straightforward to check that $F_{\mathrm{R}}^{ab}F_{\mathrm{R}ab} = 0$, and $\epsilon_{abcd}F_{\mathrm{R}}^{cd}F_{\mathrm{R}}^{ab} = 0$. That is, the radiative part of an LWF is a null field and thus it may be algebraically classified as in Steward (1990). We might use this result to extend our analogy further and relating F_{R}^{ab} to the type-N RT solutions (Kramer *et al* 1980). We have to keep in mind though that, strictly, we have not Petrov-classified the radiative part of a LWF and thus calling the LWF type N to extend the analogy is pushing the terminology too far.

Summarizing, the bounded part of an LWF follows the path II → D → O in the Penrose diagram as the motion of the point-charge changes from arbitrary, to constant, to the special case of vanishing 4-acceleration with vanishing 3-velocity; analogous behaviour occurs on specializing RT solutions (Kramer *et al* 1980). The radiative part of the LWF is always (in some cases trivially) a null electromagnetic field.

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